

On a connection between Stein characterizations and Fisher information

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Abstract: We generalize the so-called *density approach* to Stein characterizations of probability distributions. We prove an elementary factorization property of the resulting Stein operator in terms of a *generalized (standardized) score function*. We use this result to connect Stein characterizations with information distances such as the *generalized (standardized) Fisher information*.

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1. Foreword

In recent years a number of authors have noted how Charles Stein’s characterization of the Gaussian (see [11]) and the so-called “magic factors” crop up in matters related to information theory (see [5], [6], [7], [3] or [1] and the references therein). The purpose of this note is to make this connection explicit.

2. Results

We consider densities $p : \mathbb{R} \rightarrow \mathbb{R}^+$ whose support is an interval $S := S_p$ with closure $\bar{S} = [a, b]$, for some $-\infty \leq a < b \leq \infty$. Among these we denote by \mathcal{G} the collection of densities which are (strongly) differentiable at every point in the interior of their support.

Definition 2.1. Fix $p \in \mathcal{G}$ with support S and define $\mathcal{F}(p)$ the collection of test functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the mapping $x \mapsto f(x)p(x)$ is bounded on \mathbb{R} and strongly differentiable on the interior of S .

Take a real bounded function h with support S , and suppose that h is (strongly) differentiable on the interior of S . Then h can be written as $\tilde{h}\mathbb{I}_S$ with

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\tilde{h} a differentiable continuation of h on \mathbb{R} . In the sequel we will write $\partial_y h(y)|_{y=x}$ for the differential in the sense of distributions of h evaluated at x , so that $\partial_y h(y)|_{y=x} = (\tilde{h})'(x)\mathbb{I}_S(x) + \tilde{h}(x)(\delta_{\{x=a\}} - \delta_{\{x=b\}})$ where δ represents a Dirac delta.

Definition 2.2. Let \mathbb{R}^* be the collection of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$. We define the (location-based) Stein operator as the operator $\mathcal{T} : \mathbb{R}^* \times \mathcal{G} \rightarrow \mathbb{R}^* : (f, p) \mapsto \mathcal{T}(f, p)$ such that

$$\mathcal{T}(f, p) : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \frac{\partial_y(f(y)p(y))|_{y=x}}{p(x)} \quad (2.1)$$

for all f for which the differential (in the sense of distributions) exists.

Remark 2.1. The terminology “location-based” Stein operator is inherited from our parametric approach to Stein characterizations (see [8]), where a much more general characterization result is proposed.

To avoid ambiguities related to division by 0, throughout this paper we adopt the convention that, whenever an expression involves the division by an indicator function \mathbb{I}_A for some measurable set A , we are multiplying the expression by the said indicator function. This convention ensures that for $p \in \mathcal{G}$ and $f \in \mathcal{F}(p)$ and for any continuous random variable X , the quantity $\mathcal{T}(f, p)(X)$ is well-defined. We further draw the reader's attention to the fact that, in particular, ratios $p(x)/p(x)$ do not necessarily simplify to 1.

Example 2.1. It is perhaps informative to see how Definitions 2.1 and 2.2 spell out for different explicit choices of target densities.

1. If $p = \phi$, the standard Gaussian, then $\mathcal{F}(\phi)$ contains the set of all differentiable bounded functions and

$$\mathcal{T}(f, \phi)(x) = f'(x) - xf(x),$$

which is Stein's well-known operator for characterizing the Gaussian.

2. If $p(x) = e^{-x}\mathbb{I}_{[0, \infty)}(x)$, the exponential Exp , then (abusing notations) $\mathcal{F}(Exp)$ contains the set of all differentiable bounded functions and

$$\mathcal{T}(f, Exp)(x) = (f'(x) - f(x) + f(x)\delta_{\{x=0\}})\mathbb{I}_{[0, \infty)}(x).$$

3. If $p(x) = \mathbb{I}_{[0, 1]}(x)$, the standard uniform $U(0, 1)$, then $\mathcal{F}(U(0, 1))$ contains the set of all differentiable bounded functions and

$$\mathcal{T}(f, U(0, 1))(x) = (f'(x) + f(x)(\delta_{\{x=0\}} - \delta_{\{x=1\}}))\mathbb{I}_{[0, 1]}(x).$$

4. If $p(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbb{I}_{(-2, 2)}(x)$, Wigner's semicircle law SC , then $\mathcal{F}(SC)$ contains the set of all functions of the form $f(x) = f_0(x)(4-x^2)$ for some bounded differentiable f_0 and, for these f , the operator becomes

$$\mathcal{T}(f, SC)(x) = ((4-x^2)f_0'(x) - 3xf_0(x))\mathbb{I}_{(-2, 2)}(x).$$

5. If $f(x) = \frac{1}{\pi\sqrt{x(1-x)}}\mathbb{I}_{(0,1)}(x)$, the arcsine distribution AS , then $\mathcal{F}(AS)$ contains the collection of all functions of the form $f(x) = f_0(x)\sqrt{x(1-x)}$ for some bounded differentiable f_0 and, for these f , the operator becomes

$$\mathcal{T}(f, AS)(x) = \sqrt{x(1-x)}f'_0(x)\mathbb{I}_{(0,1)}(x).$$

6. If $p(x)$ is a member of Pearson's family of distributions and thus satisfies

$$(s(x)p(x))' = \tau(x)p(x)$$

for τ a polynomial of exact degree one and s a polynomial of degree at most two, then, abusing notations one last time, we easily see that $\mathcal{F}(P(s, \tau))$ contains the set of all functions of the form $f(x) = f_0(x)s(x)$ for f_0 bounded, differentiable such that $f(a^+) = f(b^-) = 0$ and, for these f , the operator becomes

$$\mathcal{T}(f, P(s, \tau))(x) = (s(x)f'_0(x) + \tau(x)f_0(x))\mathbb{I}_S(x).$$

The first three operators are well-known and can be found, for instance, in [12]. The fourth example can be found in [4]. The last example comes from [9].

We are now ready to state and prove our first main result.

Theorem 2.1 (Density approach). *Let $p \in \mathcal{G}$ with support S , and take $Z \sim p$. Let $\mathcal{F}(p)$ be as in Definition 2.1 and \mathcal{T} as in Definition 2.2. Let X be a real-valued continuous random variable.*

- (1) *If $X \stackrel{\mathcal{L}}{=} Z$ then $\mathbb{E}[\mathcal{T}(f, p)(X)] = 0$ for all $f \in \mathcal{F}(p)$.*
(2) *If $\mathbb{E}[\mathcal{T}(f, p)(X)] = 0$ for all $f \in \mathcal{F}(p)$, then $X | X \in S \stackrel{\mathcal{L}}{=} Z$.*

Proof. To see (1), note that the hypotheses on f and p guarantee that we have $\mathbb{E}[\mathcal{T}(f, p)(Z)] = [f(y)p(y)]_a^b + f(a^+)p(a^+) - f(b^-)p(b^-) = 0$. To see (2), consider for $z \in \mathbb{R}$ the functions f_z^p defined through

$$f_z^p : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \frac{1}{p(x)} \int_a^x l_z(u)p(u)du$$

with $l_z(u) := (\mathbb{I}_{(-\infty, z]}(u) - \mathbb{P}_p(X \leq z))\mathbb{I}_S(u)$ and $\mathbb{P}_p(X \leq z) := \int_{-\infty}^z p(u)du$. Clearly $f_z^p \in \mathcal{F}(p)$ for all z . Moreover we have $\partial_y(f_z^p(y)p(y))|_{y=x} = l_z(x)p(x)$ since $\int_a^c l_z(u)p(u)du = 0$ for $c = a$ and $c = b$. Therefore f_z^p satisfies, for all z , the so-called *Stein equation*

$$\mathcal{T}(f_z^p, p)(x) = l_z(x). \quad (2.2)$$

Hence we can use $\mathbb{E}[\mathcal{T}(f_z^p, p)(X)] = 0$ to deduce that $\mathbb{P}(X \in (-\infty, z] \cap S) = \mathbb{P}(Z \leq z)\mathbb{P}(X \in S)$ for all z , whence the claim. \square

Theorem 2.1 encompasses Proposition 4 in [12] and Theorem 1 in [9] and is easily shown to contain many of the other better known Stein characterizations (such as the characterization of the semi-circular in [4]). We draw the reader's

attention to the fact that our way of writing the Stein operator (2.1) also shows that all Stein equations of the form (2.2) (that is, most such equations from the literature) can be solved by simple integration. Also, the form of our operators leads directly to our second main result.

Theorem 2.2 (Factorization Theorem of Stein Operators). *Let p and q be probability density functions in \mathcal{G} sharing support S . For all $f \in \mathcal{F}(p) \cap \mathcal{F}(q)$, we have*

$$\mathcal{T}(f, p)(x) = \mathcal{T}(f, q)(x) + f(x)r(p, q)(x),$$

with

$$r(p, q)(x) := \frac{p'(x)}{p(x)} - \frac{q'(x)}{q(x)} + (\delta_{\{x=a\}} - \delta_{\{x=b\}})\mathbb{I}_S(x).$$

Proof. The restriction on the support of q guarantees that we have $f(y)p(y) = f(y)q(y)p(y)/q(y)$ for any real-valued function f . We can therefore write

$$\begin{aligned} \mathcal{T}(f, p)(x) &= \frac{\partial_y(f(y)q(y)p(y)/q(y))|_{y=x}}{p(x)} \\ &= \frac{\partial_y(f(y)q(y))|_{y=x}}{p(x)} \frac{p(x)}{q(x)} + f(x)q(x) \frac{\partial_y(p(y)/q(y))|_{y=x}}{p(x)} \\ &= \mathcal{T}(f, q)(x) + f(x) \frac{q(x)}{p(x)} \partial_y(p(y)/q(y))|_{y=x}. \end{aligned}$$

The claim follows. \square

Note that, whenever $S = \mathbb{R}$ or S is an open interval, $r(p, q)$ simplifies to $p'/p - q'/q$. Now, let l be a real-valued function. In the sequel we will write $E_p[l(X)] := \int_{\mathbb{R}} l(x)p(x)dx$. Our next and final main result is immediate and hence its proof is left to the reader.

Theorem 2.3 (Stein's method and information distances). *Let p and q be probability density functions in \mathcal{G} sharing support S . Let l be a real-valued function such that $E_p[l(X)]$ and $E_q[l(X)]$ exist. Define f_l^p to be the solution of the Stein equation*

$$\mathcal{T}(f, p)(x) = (l(x) - E_p[l(X)])\mathbb{I}_S(x) \quad (2.3)$$

and suppose that $f_l^p \in \mathcal{F}(q)$. Then

$$E_q[l(X)] - E_p[l(X)] = E_q[f_l^p(X)r(p, q)(X)]. \quad (2.4)$$

Whenever p is well-behaved, the solutions to (2.3) are of the well-known form

$$f_l^p : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \frac{1}{p(x)} \int_a^x (l(u) - E_p[l(X)])p(u)du. \quad (2.5)$$

In cases such as the SC or the AS, the form of this solution (expressed in terms of f_0 instead of f) is slightly different but easily provided, see Example 2.1 or equations (18) and (19) in Proposition 1 of [9].

In all explicit instances covered in Example 2.1, the condition that $f_l^p \in \mathcal{F}(q)$ is trivially verified (see page 4 in [2] for the Gaussian). Under moment conditions on p , Schoutens shows that members of the Pearson family satisfy this assumption as well (see [9], Lemma 1).

3. Application

Applying Hölder's inequality to (2.4) shows that, under the same conditions,

$$|\mathbb{E}_q[l(X)] - \mathbb{E}_p[l(X)]| \leq \kappa_l^p \sqrt{\mathbb{E}_q[(r(p, q)(X))^2]}, \quad (3.1)$$

with

$$\kappa_l^p = \sqrt{\mathbb{E}_q[(f_l^p(X))^2]}.$$

Equation (3.1) provides some form of universal bound on differences of expectations in terms of what can be likened to a *generalized (standardized) Fisher information distance*

$$\mathcal{J}(p, q) = \mathbb{E}_q[(r(p, q)(X))^2]$$

(the terminology and notations are borrowed from [1]). Note how, for instance, taking $p = \phi$ the standard Gaussian density yields the *Fisher information distance* studied, e.g., in [6].

Theorem 2.3 also provides a bound on any probability metric which can be written as

$$d_{\mathcal{H}}(p, q) = \sup_{h \in \mathcal{H}} |\mathbb{E}_q[h(X)] - \mathbb{E}_p[h(X)]| \quad (3.2)$$

for some class of functions \mathcal{H} . The *Kolmogorov*, *Wasserstein* and *total variation* distances, to cite but these, can all be written in this form.

Specifying the target as well as the class \mathcal{H} yields the following immediate corollaries.

Corollary 3.1. *Let p and q be probability densities with support $S \subseteq \mathbb{R}$ satisfying the hypotheses in Theorem 2.3. Then there exist constants $\kappa_1 := \kappa_1(p, q)$ and $\kappa_2 := \kappa_2(p, q)$ such that*

$$\int |p(u) - q(u)| du \leq \kappa_1 \sqrt{\mathcal{J}(p, q)}$$

and

$$\sup_{x \in \mathbb{R}} |p(x) - q(x)| \leq \kappa_2 \sqrt{\mathcal{J}(p, q)}.$$

Proof. Take $l(u) = \mathbb{I}_{\{p(u) \leq q(u)\}} - \mathbb{I}_{\{p(u) \geq q(u)\}}$. Using (2.4) with this choice of l and applying Hölder's inequality, one readily sees that there exists a constant $\kappa_1 > 0$ such that

$$\int |p(x) - q(x)| dx \leq \kappa_1 \sqrt{\mathcal{J}(p, q)}$$

where $\kappa_1 = \sqrt{\mathbb{E}_q[(f_l^p(X))^2]}$.

Regarding the second inequality first note that, whenever $x \in S^c$, $|p(x) - q(x)| = 0$, hence we can concentrate on the supremum over the support S . Now choose $l(u) = \delta_{\{x=u\}}$ the Dirac delta function in $x \in S$. For this choice of l we obtain after some computations

$$|q(x) - p(x)| \leq p(x) \sqrt{\mathbb{E}_q \left[\left(\mathbb{I}_{[x,b)}(X) - P(X) \right)^2 / (p(X))^2 \right]} \sqrt{\mathcal{J}(p, q)},$$

where P is the cumulative distribution function of the density p (for which evidently $P(a) = 0$). Taking the supremum yields the second constant κ_2 . \square

We conclude this paper with a computation of bounds on the constants κ_1 and κ_2 for various examples. While these are somewhat related to the so-called “magic factors” appearing in the literature on Stein’s method, the technique we employ to bound them is different and, we believe, of independent interest. To the best of our knowledge, such bounds were first obtained in [10] for Gaussian target only. Shimizu’s results were later improved and extended in [5] and [6]. We recover in Corollary 3.2 below the best known values for κ_1 and our bound for κ_2 yields a significant improvement. We stress the fact that the results available in the literature only concern a Gaussian target, whereas our approach allows to obtain such relationships for virtually any target distribution. Further explorations of the consequences of Theorem 2.3 also show that it is possible to relate Stein characterizations with other (pseudo-)metrics than those of the form (3.2), such as, e.g., *Kullback-Leibler divergence* or *relative entropy* (see [5]).

Corollary 3.2.

1. If p is the exponential distribution with rate 1, then $\kappa_1 \leq 1$.
2. If $p = \phi$ is the standard normal distribution, then $\kappa_1 \leq \sqrt{2}$.
3. If p is proportional to $e^{-x^4/12}$, then $\kappa_1 \leq \sqrt{2\sqrt{2}}$.

In all three cases we have $\kappa_2 \leq 1$.

Proof of the constants κ_1 . Take $l(u) = \mathbb{I}_{\{p(u) \leq q(u)\}} - \mathbb{I}_{\{p(u) \geq q(u)\}}$. Using (2.5) and the fact that $\int_a^b (l(u) - \mathbb{E}_p[l(X)])p(u)du = 0$, we obtain that

$$\begin{aligned} f_l^p(x) &= -\frac{1}{p(x)} \int_x^b (l(u) - \mathbb{E}_p[l(X)])p(u)du \\ &= -\frac{2}{p(x)} \int_x^b (\mathbb{I}_{\{p(u) \leq q(u)\}} - \mathbb{P}_p(p(X) \leq q(X)))p(u)du \\ &=: \frac{2}{p(x)} \int_x^b h(u)p(u)du, \end{aligned}$$

where $\mathbb{P}_p(X \in A) = \int_A p(u)du$ for some set A . Let $p(x) = e^{-x}\mathbb{I}_{[0,\infty)}(x)$, the density of an exponential-1 random variable (in other words, $a = 0$ and $b = \infty$).

Recall that, in this case, the support of f_l^p is a subset of \mathbb{R}^+ . Then we can write

$$\begin{aligned}\kappa_1 &:= \mathbb{E}_q[(f_l^p(X))^2] = 4 \int_0^\infty q(x) e^{2x} \left(\int_x^\infty h(u) e^{-u} du \right)^2 dx \\ &\leq 4 \int_0^\infty q(x) e^{2x} \left(\int_x^\infty h^2(u) e^{-2u} du \right) dx \\ &\leq \frac{4}{2} \int_0^\infty q(x) e^{2x} \left(\int_{2x}^\infty h^2\left(\frac{u}{2}\right) e^{-u} du \right) dx,\end{aligned}$$

where the first inequality follows from Jensen and the second inequality from a simple change of variables. Applying Hölder's inequality and again changing variables in the above yields

$$\begin{aligned}\kappa_1 &\leq \frac{4}{2} \sqrt{\int_0^\infty q(x) dx} \sqrt{\int_0^\infty q(x) e^{4x} \left(\int_{2x}^\infty h^2\left(\frac{u}{2}\right) e^{-u} du \right)^2 dx} \\ &\leq \frac{4}{2^{1+\frac{1}{2}}} \left(\int_0^\infty q(x) e^{4x} \left(\int_{4x}^\infty h^4\left(\frac{u}{4}\right) e^{-u} du \right) dx \right)^{1/2},\end{aligned}$$

where $\int_0^\infty q(x) dx = 1$ by our assumption that p and q share the same support. Iterating this procedure $m \in \mathbb{N}$ times, we obtain

$$\kappa_1 \leq \frac{4}{2^{M(m)}} \left(\int_0^\infty q(x) e^{2^{m+1}x} \left(\int_{2^{m+1}x}^\infty h^{2^{m+1}}\left(\frac{u}{2^{m+1}}\right) e^{-u} du \right) dx \right)^{1/2^m},$$

where $M(m) = 1 + \frac{1}{2} + \dots + \frac{1}{2^m}$. Now note that, for each $m \geq 0$, we have $0 \leq h^{2^{m+1}}(u/2^{m+1}) \leq 1$. Hence

$$\int_{2^{m+1}x}^\infty h^{2^{m+1}}\left(\frac{u}{2^{m+1}}\right) e^{-u} du \leq e^{-2^{m+1}x}.$$

Since $M(m) \rightarrow 2$, the result follows.

If the support of p (and hence also of q) is the real line, we use similarly as above the identity $\int_{-\infty}^\infty (l(u) - \mathbb{E}_p[l(X)]) p(u) du = 0$ to write, equivalently,

$$f_l^p(x) = \frac{2}{p(x)} \int_x^\infty h(u) p(u) du = -\frac{2}{p(x)} \int_{-\infty}^x h(u) p(u) du.$$

This yields

$$\begin{aligned}\mathbb{E}_q[(f_l^p(X))^2] &= 4 \int_{-\infty}^\infty q(x) \left(\frac{1}{p(x)} \int_x^\infty h(u) p(u) du \right)^2 dx \\ &= 4 \int_{-\infty}^0 q(x) \left(\frac{1}{p(x)} \int_{-\infty}^x h(u) p(u) du \right)^2 dx \\ &\quad + 4 \int_0^\infty q(x) \left(\frac{1}{p(x)} \int_x^\infty h(u) p(u) du \right)^2 dx.\end{aligned}$$

Setting $p(x) = (2\pi)^{-1/2}e^{-x^2/2}$ we get by Jensen's inequality

$$\begin{aligned} \mathbb{E}_q[(f_l^p(X))^2] &\leq 4 \int_{-\infty}^0 q(x) \left(e^{x^2} \int_{-\infty}^x h^2(u) e^{-u^2} du \right) dx \\ &\quad + 4 \int_0^{\infty} q(x) \left(e^{x^2} \int_x^{\infty} h^2(u) e^{-u^2} du \right) dx =: I^- + I^+. \end{aligned}$$

Both integrals above can be tackled in the same way as for the exponential distribution. Consider, for instance, I^- for which we can write (thanks to a simple change of variables)

$$\begin{aligned} I^- &= 4 \int_{-\infty}^0 q(x) \left(e^{x^2} \int_{-\infty}^x h^2(u) e^{-u^2} du \right) dx \\ &= \frac{4}{\sqrt{2}} \int_{-\infty}^0 q(x) \left(e^{x^2} \int_{-\infty}^{\sqrt{2}x} h^2(u/\sqrt{2}) e^{-u^2/2} du \right) dx. \end{aligned}$$

Now apply Hölder's inequality to get

$$\begin{aligned} I^- &\leq \frac{4\sqrt{p}}{\sqrt{2}} \sqrt{\int_{-\infty}^0 q(x) \left(e^{x^2} \int_{-\infty}^{\sqrt{2}x} h^2(u/\sqrt{2}) e^{-u^2/2} du \right)^2 dx} \\ &\leq \frac{4\sqrt{p}}{\sqrt{2}} \sqrt{\int_{-\infty}^0 q(x) \left(e^{2x^2} \int_{-\infty}^{\sqrt{2}x} h^4(u/\sqrt{2}) e^{-u^2} du \right) dx}, \end{aligned}$$

where $p = P_q(X < 0)$. Changing variables once more yields

$$I^- \leq I_1^-$$

with

$$I_1^- = \frac{4p^{\frac{1}{2}}}{2^{\frac{1}{2}+1}} \left(\int_{-\infty}^0 q(x) \left(e^{2x^2} \int_{-\infty}^{(\sqrt{2})^2 x} h^4 \left(\frac{u}{(\sqrt{2})^2} \right) e^{-u^2/2} du \right) dx \right)^{\frac{1}{2}}.$$

Iterating this procedure $m \in \mathbb{N}$ times we deduce

$$I^- \leq I_1^- \leq \dots \leq I_m^-$$

with I_m^- given by

$$\frac{4p^{N(m)}}{2^{N(m)+1}} \left(\int_{-\infty}^0 q(x) \left(e^{2^m x^2} \int_{-\infty}^{(\sqrt{2})^{m+1} x} h^{2^{m+1}} \left(\frac{u}{(\sqrt{2})^{m+1}} \right) e^{-u^2/2} du \right) dx \right)^{\frac{1}{2^m}}$$

where we set $N(m) = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^m} (= M(m) - 1)$. For every m we have $0 \leq h^{2^{m+1}}(u/(\sqrt{2})^{m+1}) \leq 1$ and

$$\int_{-\infty}^0 q(x) \left(e^{2^m x^2} \int_{-\infty}^{(\sqrt{2})^{m+1} x} e^{-u^2/2} du \right) dx \leq P_q(X < 0) \sqrt{\frac{\pi}{2}}.$$

Therefore

$$I_m^- \leq \frac{4}{2^{N(m+1)}} (\mathbb{P}_q(X < 0))^{N(m)} \left(\mathbb{P}_q(X < 0) \sqrt{\frac{\pi}{2}} \right)^{1/2^m}.$$

Since $N(m) \rightarrow 1$ as $m \rightarrow \infty$, we conclude

$$I^- \leq 2 \mathbb{P}_q(X < 0).$$

One can similarly show that $I^+ \leq 2 \mathbb{P}_q(X > 0)$, and the result follows.

The computations for densities proportional to $e^{-x^4/12}$ are similar and are left to the reader. \square

Proof of the constants κ_2 . Let $p(x) = e^{-x} \mathbb{I}_{[0, \infty)}(x)$, which readily implies $P(x) = (1 - e^{-x}) \mathbb{I}_{[0, \infty)}(x)$. This leads to

$$\begin{aligned} & \mathbb{E}_q \left[(\mathbb{I}_{[x, \infty)}(X) - P(X))^2 / (p(X))^2 \right] \\ &= \int_0^\infty q(y) e^{2y} (\mathbb{I}_{[x, \infty)}(y) - 1 + e^{-y})^2 dy \\ &= \int_0^x q(y) e^{2y} (1 - e^{-y})^2 dy + \int_x^\infty q(y) e^{2y} e^{-2y} dy \\ &= \int_0^x q(y) e^{2y} (1 - 2e^{-y}) dy + \int_0^x q(y) dy + \int_x^\infty q(y) dy \\ &\leq 1 + e^{2x} (1 - 2e^{-x}) \mathbb{P}_q(X \leq x), \end{aligned}$$

since $e^{2y}(1 - 2e^{-y})$ is a monotone increasing function on \mathbb{R}^+ . This immediately yields

$$\kappa_2 \leq \sup_{x \geq 0} \left(e^{-x} \sqrt{1 + e^{2x} (1 - 2e^{-x}) \mathbb{P}_q(X \leq x)} \right),$$

a quantity which can be bounded by 1.

Now let $p(x) = (2\pi)^{-1/2} e^{-x^2}$ and $P(x) = \Phi(x)$, the cumulative distribution function of the standard normal distribution. Similarly as for the exponential, we have

$$\begin{aligned} & \mathbb{E}_q [(\mathbb{I}_{[x, \infty)}(X) - P(X))^2 / (p(X))^2] \\ &= 2\pi \int_{-\infty}^\infty q(y) e^{y^2} (\mathbb{I}_{[x, \infty)}(y) - \Phi(y))^2 dy \\ &= 2\pi \int_{-\infty}^x q(y) e^{y^2} (\Phi(y))^2 dy + 2\pi \int_x^\infty q(y) e^{y^2} (1 - \Phi(y))^2 dy \\ &\leq 2\pi e^{x^2} (\Phi(x))^2 \int_{-\infty}^x q(y) dy + 2\pi e^{x^2} (1 - \Phi(x))^2 \int_x^\infty q(y) dy \\ &= 2\pi e^{x^2} (\Phi(x))^2 + 2\pi e^{x^2} (1 - 2\Phi(x)) \mathbb{P}_q(X \geq x). \end{aligned}$$

This again directly leads to

$$\begin{aligned}\kappa_2 &\leq \sup_{x \in \mathbb{R}} \left((2\pi)^{-1/2} e^{-x^2/2} \sqrt{2\pi e^{x^2} ((\Phi(x))^2 + (1 - 2\Phi(x))P_q(X \geq x))} \right) \\ &= \sup_{x \in \mathbb{R}} \left(\sqrt{(\Phi(x))^2 + (1 - 2\Phi(x))P_q(X \geq x)} \right),\end{aligned}$$

a quantity which can be shown to equal 1.

The computations for densities proportional to $e^{-x^4/12}$ are similar and are left to the reader. \square

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